Technical Communique
On minimal-order stabilization of minimum phase plants

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Abstract
In this note, the problem of minimal-order stabilization in the case where the plant is minimum phase is studied. A low bound on the order of stabilizers is derived and a set of minimal-order stabilizers are characterized. The low bound is related to the number and location of the plant’s unstable and lightly damped poles and the number of zeros. How to construct a minimal-order or low-order stabilizer for a general case is also discussed and the algorithm is provided. Numerical examples are given to illustrate the proposed method. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction
The stabilization is a basic problem that control theory attempts to solve. Numerous papers have been devoted to it (Youla, Bongiorno, & Lu, 1974; Kwakernaak, 1985; Fu, Olbrot, & Polis, 1989; Argoun, 1990). However, existing solutions to the problem usually generate unnecessary high-order controllers. As low-order controllers are normally preferred to high-order ones for ease of implementation, verification, computation, and maintenance, the order, i.e., the McMillan degree of a controller, becomes a research topic of great practical significance. Unfortunately, simple problems cannot always be simply solved. For being open for decades, the low-order stabilization problem, especially, the minimal-order stabilization problem, does not seem to have a simple solution.

In this note, we focus on the problem of how to characterize a set of minimal-order stabilizers for minimum phase plants, which gives a partial solution to the original problem.

The problem of minimal-order or low-order stabilizers has already received much attention. For example, it is shown by Smith and Sondergeld (1986) and Smith (1986) that in general the minimal order of a stabilizing stable controller may not be bounded in terms of the plant order. The bound cannot be predicted except for very special cases. In Linnemann (1988), an algebraic condition is given under which a single loop plant of order $n$ can be stabilized by a controller of order less than $n - 1$. Keel and Bhattacharyya (1990) present an indirect algorithm in the state space domain for designing low-order stabilizers. Gu, Choi, and Postlethwaite (1993) also describe a method for designing low-order stabilizers. The last results on this line are Wang, Lee, and He (1997a) and Wang, Lee, and He (1997b). They derive a lower bound on the order of stabilizers for an all-pole plant. The bound is related to the number and location of the plant’s unstable and lightly damped poles.

This note is organized as follows. In Section 2, the minimal order for stabilizers of minimum phase plants is investigated. An explicit characterization is provided. In Section 3, how to construct a minimal-order or low-order stabilizer is discussed and the proposed method is illustrated and verified by several numerical examples. Finally, conclusions are given in Section 4.

2. Estimation of the minimal order
In Wang et al. (1997a), the minimal order has been estimated for all-pole plants (i.e., plants without zeros). It is the purpose of this complementary note to extend the result to minimum phase plants (i.e., plants with left half plane zeros). Consider the unity feedback control system shown in

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Fig. 1. Unity feedback control system.

Fig. 1, where \( G(s) \) is the minimum phase plant and \( C(s) \) is the controller. Assume that

\[
G(s) = \frac{N_c(s)}{M_c(s)},
\]

where \( M_c(s) \) and \( N_c(s) \) are coprime polynomials, \( n = \text{deg}\{M_c(s)\} > q = \text{deg}\{N_c(s)\} \), and \( N_c(s) \) is Hurwitz.

**Theorem 1.** If the plant has \( \gamma \) poles, \( \lambda_1, \lambda_2, \ldots, \lambda_\gamma, \gamma \leq n \), such that they satisfy

\[
0 \leq \frac{\lambda_i}{\lambda_j}, \frac{\lambda_i}{\lambda_k}, \ldots, \frac{\lambda_i}{\lambda_j} \leq \gamma, \quad k = 1, 2, \ldots, n - 1,
\]

then the minimal order of stabilizers is equal to or greater than \( \gamma - q - 1 \).

**Proof.** Assume that the stabilizer is expressed as

\[
C(s) = \frac{N_c(s)}{M_c(s)},
\]

where \( M_c(s) \) and \( N_c(s) \) are coprime polynomials, and \( m = \text{deg}\{M_c(s)\} \geq \text{deg}\{N_c(s)\} \). The resultant closed loop characteristic polynomial is

\[
h(s) = M_G(s)C(s) + N_G(s)N_C(s).
\]

The closed loop system is internally stable if and only if \( h(s) \) is Hurwitz (Rosenbrock, 1974). Let \( \lambda_i, i = 1, 2, \ldots, \gamma \), be distinct poles of \( G(s) \), and

\[
N_G(s)N_C(s) = a_0 + a_1s + \cdots + a_{m+q}s^{m+q},
\]

\[
h(s) = h_0 + h_1s + \cdots + h_{n+m}s^{n+m}.
\]

Without loss of generality, it is assumed that \( h_i > 0 \), \( i = 0, 1, 2, \ldots, n + m \). Since \( M_G(\lambda_i) = 0 \), we have

\[
V_{\gamma}^{m+q+1} N = V_{\gamma}^{m+1} H,
\]

where

\[
N = (a_0 \quad a_1 \quad \cdots \quad a_{m+q})^T,
\]

\[
H = (h_0 \quad h_1 \quad \cdots \quad h_{n+m})^T,
\]

\[
V_{\gamma} = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_\gamma & \lambda_\gamma^2 & \cdots & \lambda_\gamma^{k-1} \end{pmatrix}.
\]

Because the order of stabilizers is always equal to or greater than zero, we only need to consider \( \gamma + 1 \). Suppose now that \( m + q + 1 < \gamma \). It is well known that \( V_{\gamma}^{m+q+1} \) is a Vandermonde matrix and nonsingular for distinct \( \lambda_i \). Multiplying both sides of (7) by \( (V_{\gamma}^{m+q+1})^{-1} \) and using a suitable partition of \( V_{\gamma}^{m+q+1} \) and \( H \) yields

\[
(I_{m+q+1} \quad 0) N = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{\gamma-1} \end{pmatrix} + (V_{\gamma}^{m+q+1})^{-1} \begin{pmatrix} \lambda_1 \lambda_1^{\gamma+1} & \cdots & \lambda_1^{\gamma+m} \\ \lambda_2 \lambda_2^{\gamma+1} & \cdots & \lambda_2^{\gamma+m} \\ \vdots & \vdots & \vdots \\ \lambda_\gamma \lambda_\gamma^{\gamma+1} & \cdots & \lambda_\gamma^{\gamma+m} \end{pmatrix} \times \begin{pmatrix} h_1 \\ h_{\gamma+1} \\ \vdots \\ h_{n+m} \end{pmatrix}.
\]

At this stage, if we can show that \( m + q + 1 < \gamma \) will lead to a contradiction, then \( m + q + 1 = \gamma \) is true. The procedure has been given by Wang et al. (1997a) and Morris (1983) and is briefly stated here.

Let \( c_{i,j} \) be the cofactor of the element in \( V_{\gamma}^{m+q+1} \), and \( \det(.) \) stand for a determinant. Consider the last row, \( v_k = c_{k,\gamma}/\det(V_{\gamma}^{m+q+1}) \), \( k = 1, 2, \ldots, \gamma \), of the matrix \((V_{\gamma}^{m+q+1})^{-1}\). The last row of (8) thus becomes

\[
h_{\gamma-1} + \sum_{k=1}^{n+m-\gamma+1} \frac{\sum_{i=1}^{\gamma} c_{i,k} \lambda_i^{\gamma+k-1}}{\det(V_{\gamma}^{m+q+1})} h_{\gamma+k-1} = 0.
\]

Let

\[
T_{\gamma}^{k} = \sum_{i=1}^{\gamma} c_{i,\gamma} \lambda_i^{\gamma+k-1}
\]

which will be proven by induction on \( \gamma \). Although only \( \gamma > q + 1 \) needs to be considered, we begin the procedure with \( \gamma = 2 \).

\[
T_{\gamma}^{k} = \det(V_{\gamma}^{m+q+1}) \sum_{0 \leq i_1, i_2, \ldots, i_\gamma \leq k} \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_\gamma^{i_\gamma}
\]

(11)

We now claim that

\[
T_{\gamma}^{k} = \det(V_{\gamma}^{m+q+1}) \sum_{i_1 + i_2 + \cdots + i_\gamma = k} \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_\gamma^{i_\gamma}
\]

(11)

which will be proven by induction on \( \gamma \). Although only \( \gamma > q + 1 \) needs to be considered, we begin the procedure with \( \gamma = 2 \).

\[
T_{\gamma}^{k} = \begin{pmatrix} 1 & \lambda_1^{i+k} \\ \lambda_2^{i+k} \end{pmatrix} = (\lambda_2 - \lambda_1)(\lambda_2^{i+k} - \lambda_1^{i+k})
\]
which satisfies (11). Suppose now that (11) is true for \( \gamma - 1 \), it can be proven that (11) is also true for \( \gamma \). Combined with (2), we have

\[
h_{\gamma - 1} + \sum_{k=1}^{n+m-\gamma+1} \frac{h_k}{\gamma \cdot \gamma^{k-1}} h_{\gamma+k-1} > 0
\]

which contradicts with (9). The proof is then completed for the case of distinct poles. If the plant has a multiple pole at \( \lambda_1 \), say with multiple 2, we may set \( \lambda_2 = \lambda_1 \). The condition now includes \( M_G(\lambda_1) = 0 \), \( M'_G(\lambda_1) = 0 \) and others remain the same. \( \square \)

Obviously, the result of Wang et al. (1997a) is a special case of the above theorem. Wang et al. (1997a) have shown through examples that a tight bound may be obtained by appropriate and repeated use of the theorem.

Example 1. Consider a plant

\[
G(s) = \frac{s + 1}{s^3 + s^2 - 14s - 24} = \frac{s + 1}{(s + 2)(s + 3)(s - 4)}.
\]

Choose poles at 4 and \(-2\), then \( \gamma = 2 \) and thus \( m \geq 0 \). We guess that the minimal order is zero. In fact, there do exist constant stabilizers, for example, \( C(s) = 30 \). In this case the closed loop poles are \(-0.3097 + 3.9583i\), \(-0.3097 - 3.9583i\), and \(-0.3806\).

3. Controller construction

Wang et al. (1997a) point out that (8) may be used to construct low-order stabilizers if (2) is violated and Wang et al. (1997b) also develop three methods for constructing low-order controllers. In this note, it is shown that (8) can also be used to construct a minimal-order stabilizer of minimum phase plants for a general case.

We first consider the case that the minimal order has been obtained by Theorem 1. The design procedure is as follows.

1. Let \( m = \gamma - q - 1 \). Then, (8) gives

\[
N = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{\gamma-1} \\ \vdots \\ h_{m+1} \end{pmatrix}
\]

\[
+ (V_{\gamma}^T)^{-1} \begin{pmatrix} \lambda_{1}^{\gamma} & \lambda_{2}^{\gamma} & \cdots & \lambda_{m}^{\gamma} \\ \lambda_{1}^{\gamma+1} & \lambda_{2}^{\gamma+1} & \cdots & \lambda_{m}^{\gamma+1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1}^{m} & \lambda_{2}^{m} & \cdots & \lambda_{m}^{m} \end{pmatrix} \begin{pmatrix} h_2 \\ h_3 \\ \vdots \\ h_{m+1} \end{pmatrix}.
\]

(12)

2. In light of (4), we have

\[
h(s) - N^T(1 s \cdots s^{m+q})^T = M_G(s)M'_G(s).
\]

For a Hurwitz \( h(s) \), if \( M_G(s) \) is the factor of \( h(s) - N^T(1 s \cdots s^{m+q}) \), the minimal-order stabilizer is obtained.

(3) Not always can one find a solution of the minimal-order stabilizer even when the solution exists. Then, an additional pole should be considered and one should repeat the above procedure until the stabilizer is obtained. The following theorem can be used to estimate the worst case.

Theorem 2 (Smith & Sondergeld, 1986). Any minimum phase plant can be stabilized by a controller of order \( m \leq n - q - 1 \).

When the condition of Theorem 1 is violated, we can start form Theorem 2 to derive a lower order stabilizer. The first step is to test \( m = n - q - 1 \). If this succeeds, test \( m = n - q - 2 \) until the test fails.

Example 2. Consider the plant of Example 1

\[
G(s) = \frac{s + 1}{s^3 + s^2 - 14s - 24}.
\]

Select poles at 4 and \(-2\), then \( \gamma = 2 \) and thus \( m \geq 0 \). Let \( m = 0 \), (12) gives

\[
N = \begin{pmatrix} h_0 + 8h_2 + 16h_3 \\ h_1 + 2h_2 + 12h_3 \end{pmatrix}.
\]

Then

\[
h(s) = N^T(1 s \cdots s^{m+q})^T
\]

\[
= -(8h_2 + 16h_3) - (2h_2 + 12h_3)s + h_3s^2 + h_3s^3.
\]

Obviously, if \( h_2 = h_3 \), \( M_G(s) \) is the factor of \( h(s) - N^T(1 s \cdots s^{m+q}) \). The question is whether \( h(s) = h_0 + h_1s + h_2s^2 + h_3s^3 \) is Hurwitz or not. This needs a trial and error procedure by using Routh–Hurwitz theorem or computer simulation. As a matter of fact, one can easily verify whether a polynomial is Hurwitz or not in MATLAB. \( h(s) = 6 + 16s + s^2 + s^3 \) is a required solution. The stabilizer is

\[
C(s) = \frac{N^T(1 s)^T(s + 1)}{1} = 30.
\]

If the solution is not obtained, the pole \(-3\) should be added and we may try \( m = 1 \). Certainly, the resultant stabilizer will not be the minimal order. By Theorem 2 \( m = 1 \) is also the upper bound of low-order stabilizer for the given plant. (12) gives

\[
N = \begin{pmatrix} h_0 + 24h_3 - 24h_4 \\ h_1 + 14h_3 + 10h_4 \\ h_2 - h_3 + 15h_4 \end{pmatrix}.
\]

Without loss of generality, it is assumed that a Hurwitz polynomial’s coefficients are positive. Let \( N_C(s) = b_0 + b_1s \).

We have

\[
h_0 + h_2 - h_1 = -9h_3 + 19h_4.
\]
On the other hand,
\[
h(s) - N^T (1 \ s \ \cdots \ s^{m+q})^T \\
= -24(h_3 - h_4) - (14h_3 + 10h_4)s \\
+ (h_3 - 15h_4)s^2 + h_3s^3 + h_4s^4.
\]

Theorem 2 tells us that there must exist a solution for which \( h(s) \) is Hurwitz. It can be obtained by solving a series of linear equations. Assume that \( M_C(s) = c_0 + c_1s \). We have
\[
c_1 = h_3, \\
c_0 + c_1 = h_3, \\
1 = h_3 - h_4.
\]

Therefore, \( c_0 = 1 \) and \( c_1 = h_4 \). It might as well let \( c_1 = 1 \).
\[
h(s) = h_0 + (h_0 + h_2 - 1)s + h_2s^2 + 2s^3 + s^4.
\]

Take \( h_0 = 10 \) and \( h_2 = 50 \), then \( h(s) = 10 + 59s + 50s^2 + 2s^3 + s^4 \), which is Hurwitz. The resultant stabilizer is
\[
C(s) = \frac{N^T (1 \ s \ s^2)^T / (s + 1)}{s + 1} = \frac{63s + 34}{s + 1}.
\]

4. Conclusions

In this note, the stabilization problem of minimum phase plants is formulated. The main results of the note are:

1. Based on the discussion of Wang et al. (1997a), a condition for the existence of minimal-order stabilizer of minimum phase plants has been derived and it is shown that the present condition is a natural generalization of the earlier known condition for the case of all-pole plants.

2. Based on the proving procedure of Theorem 1, a stabilizer design procedure is developed for minimum phase plants. It is shown by several numerical examples that the proving procedure provides a general way for constructing the minimal-order or low-order stabilizers.

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References


